计算机辅助几何设计
2021秋学期

Bézier Curves (continue)

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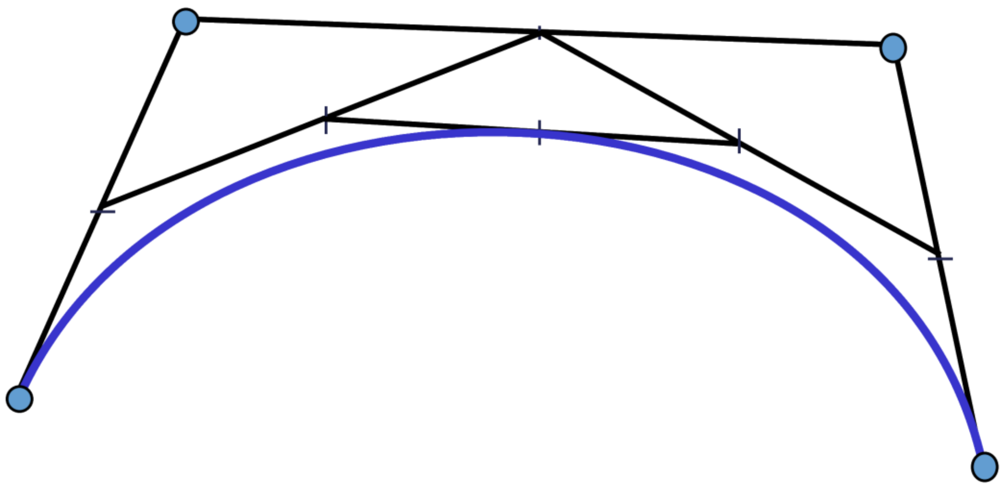
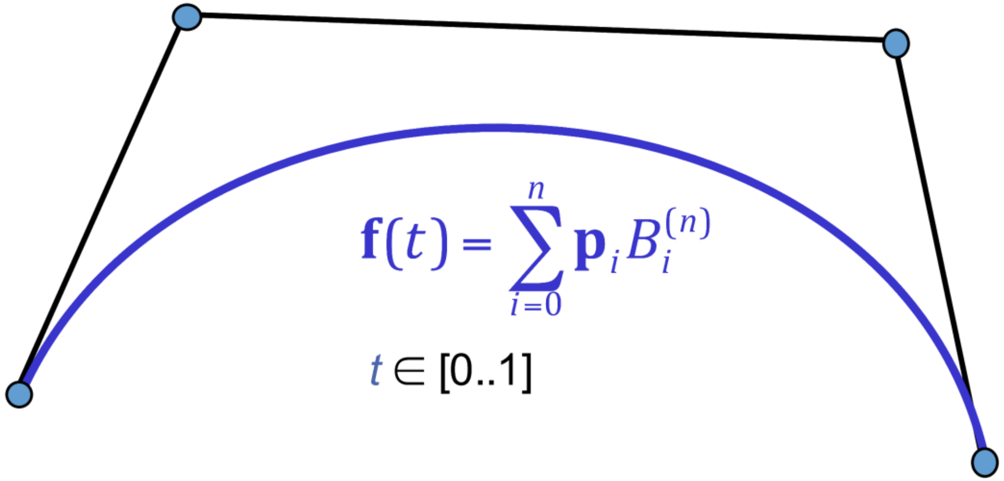
Recap

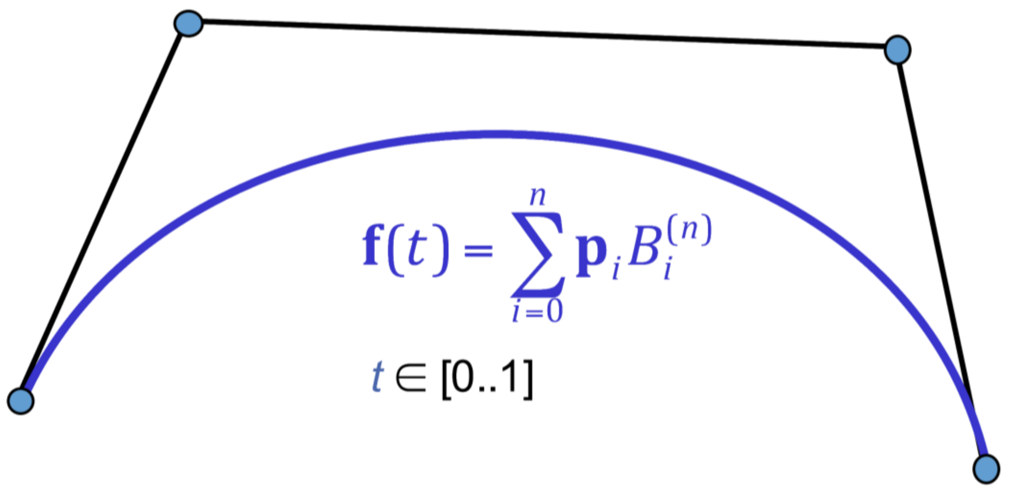


de Casteljau algorithm



Bernstein form

Recap



Bernstein form

Curve basis function control points

$$\boldsymbol{f}(t) = \sum_{i=1}^{n} B_i(t)\boldsymbol{p}_i$$

Useful properties for basis functions

- Smoothness
- Local control / support
- Affine invariance
- Convex hull property

Degree elevation

- Given: $\boldsymbol{b}_0, \cdots, \boldsymbol{b}_n \to \boldsymbol{x}(t)$
- Wanted: $\overline{\boldsymbol{b}}_0, \ldots, \overline{\boldsymbol{b}}_n, \overline{\boldsymbol{b}}_{n+1} \to \overline{\boldsymbol{x}}(t)$ with $\boldsymbol{x} = \overline{\boldsymbol{x}}$
- Solution:

Degree elevation

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- Wanted: $\overline{\boldsymbol{b}}_0, \dots, \overline{\boldsymbol{b}}_n, \overline{\boldsymbol{b}}_{n+1} \rightarrow \overline{\boldsymbol{x}}(t)$ with $\boldsymbol{x} = \overline{\boldsymbol{x}}$
- Solution:

$$\overline{\boldsymbol{b}}_0 = \boldsymbol{b}_0$$
$$\overline{\boldsymbol{b}}_{n+1} = \boldsymbol{b}_n$$
$$\overline{\boldsymbol{b}}_j = \frac{j}{n+1}\boldsymbol{b}_{j-1} + \left(1 - \frac{j}{n+1}\right)\boldsymbol{b}_j \quad \text{for } j = 1, \dots, n$$

Proof

- **Let's consider**

$$(1-t)B_i^n(t) = (1-t)\binom{n}{i}(1-t)^{n-i}t^i = \binom{n}{i}(1-t)^{n+1-i}t^i$$

$$= \frac{n+1-i}{n+1}\binom{n+1}{i}(1-t)^{n+1-i}t^i$$

$$= \frac{n+1-i}{n+1}B_i^{n+1}(t)$$

Similarly

$$tB_i^n(t) = \frac{i+1}{n+1}B_i^{n+1}(t)$$

Proof

$$\boldsymbol{f}(t) = [(1-t)+t]\boldsymbol{f}(t) = [(1-t)+t]\sum_{i=0}^{n} B_i^n(t)\boldsymbol{P}_i = \sum_{i=0}^{n}[(1-t)B_i^n(t) + tB_i^n(t)]\boldsymbol{P}_i$$

$$= \boxed{\sum_{i=0}^{n}\left[\frac{n+1-i}{n+1}B_i^{n+1}(t) + \frac{i+1}{n+1}B_{i+1}^{n+1}(t)\right]}\boldsymbol{P}_i = \sum_{i=0}^{n}\frac{n+1-i}{n+1}B_i^{n+1}(t)\boldsymbol{P}_i + \sum_{i=0}^{n}\frac{i+1}{n+1}B_{i+1}^{n+1}(t)\boldsymbol{P}_i$$

Using results from last slide

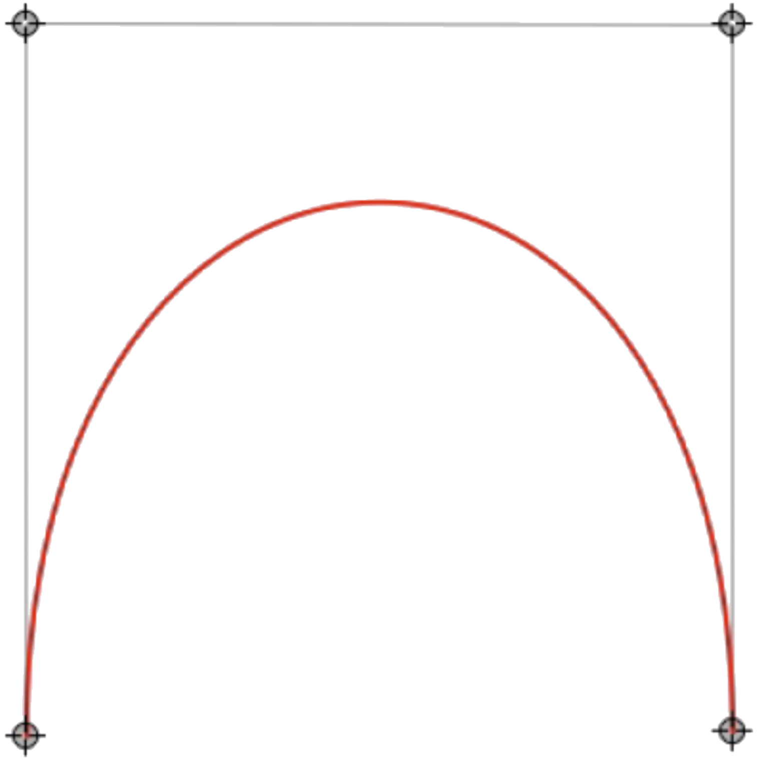
$$= \sum_{i=0}^{n}\frac{n+1-i}{n+1}B_i^{n+1}(t)\boldsymbol{P}_i + \sum_{i=1}^{n+1}\frac{i}{n+1}B_i^{n+1}(t)\boldsymbol{P}_{i-1}$$

$$= \sum_{i=0}^{n+1}\frac{n+1-i}{n+1}B_i^{n+1}(t)\boldsymbol{P}_i + \sum_{i=0}^{n+1}\frac{i}{n+1}B_i^{n+1}(t)\boldsymbol{P}_{i-1}$$

Adding null terms, $i = n+1$, $i = 0$

$$= \sum_{i=0}^{n+1}B_i^{n+1}(t)\left[\frac{n+1-i}{n+1}\boldsymbol{P}_i + \frac{i}{n+1}\boldsymbol{P}_{i-1}\right]$$

Degree elevation: Example



- $\bar{\boldsymbol{b}}_0 = \boldsymbol{b}_0$
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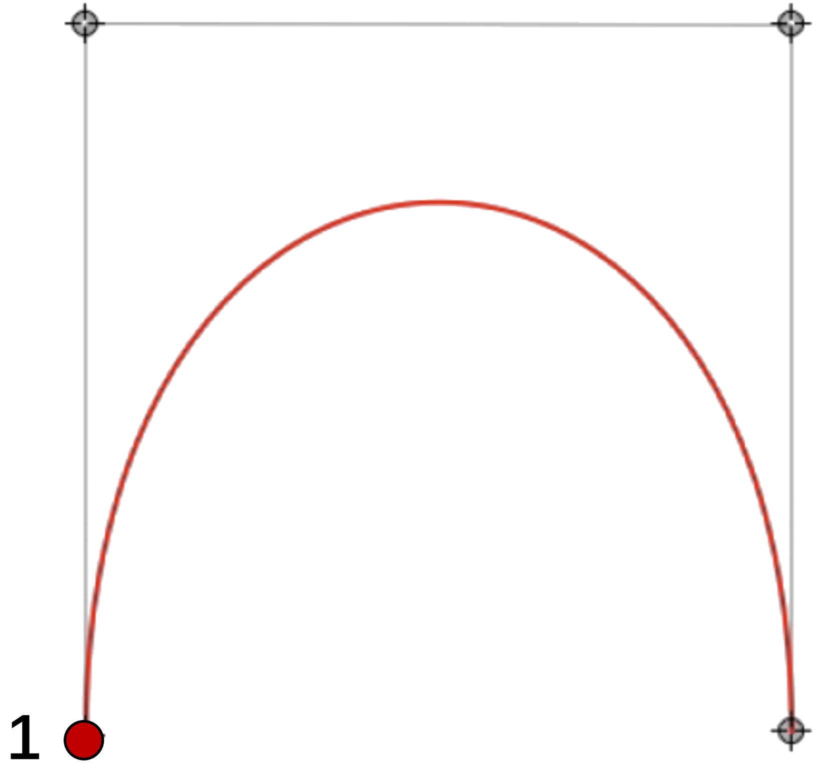
$$\bar{\boldsymbol{b}}_j = \frac{j}{n+1}\boldsymbol{b}_{j-1} + \left(1 - \frac{j}{n+1}\right)\boldsymbol{b}_j$$

$$j = 1, \dots, n$$

Degree elevation: Example



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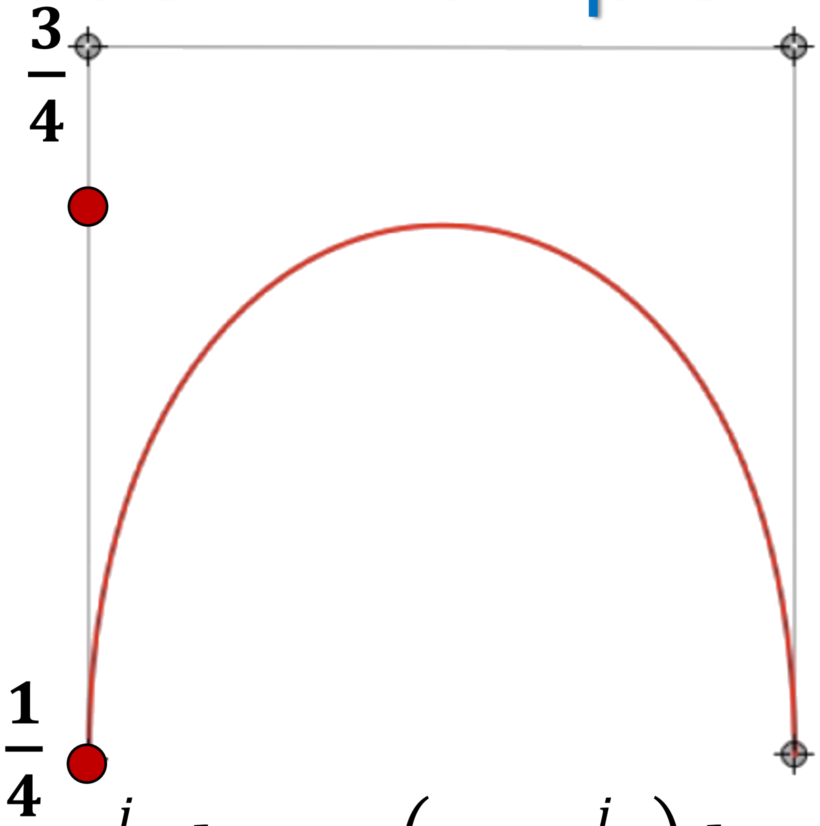

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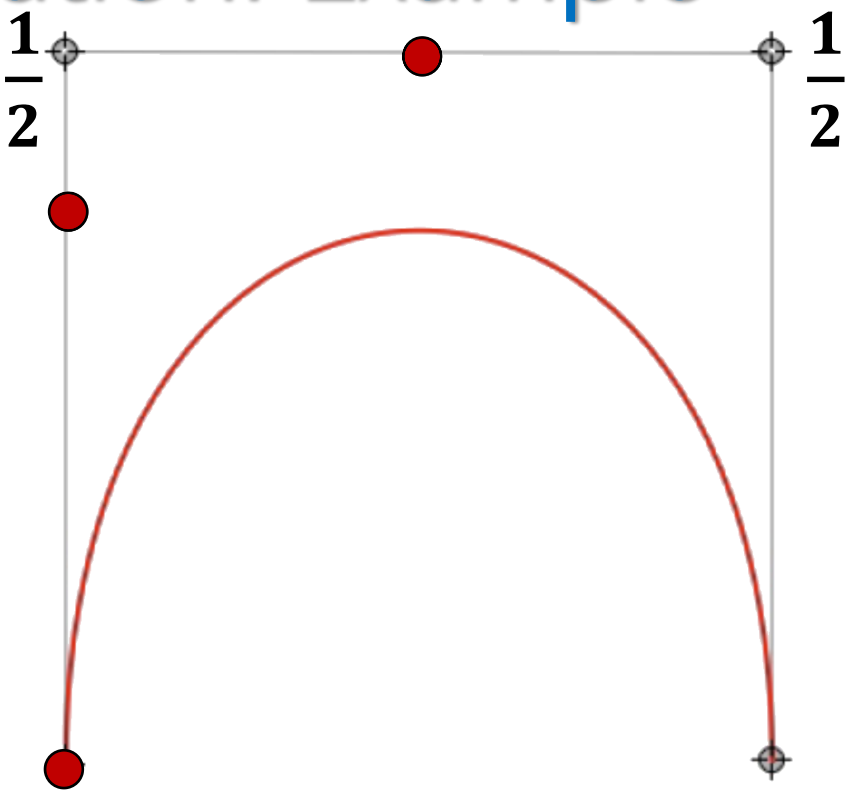

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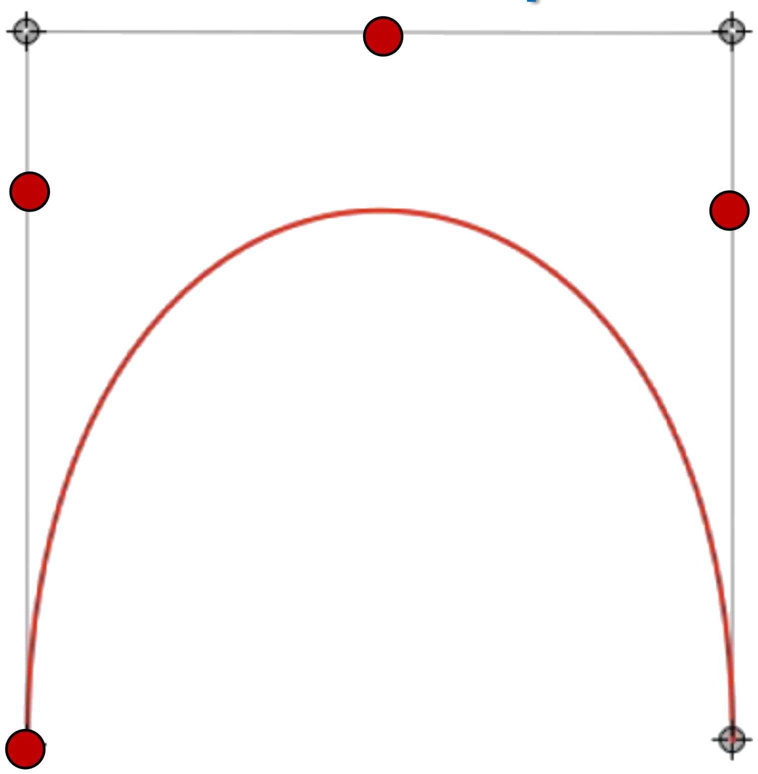

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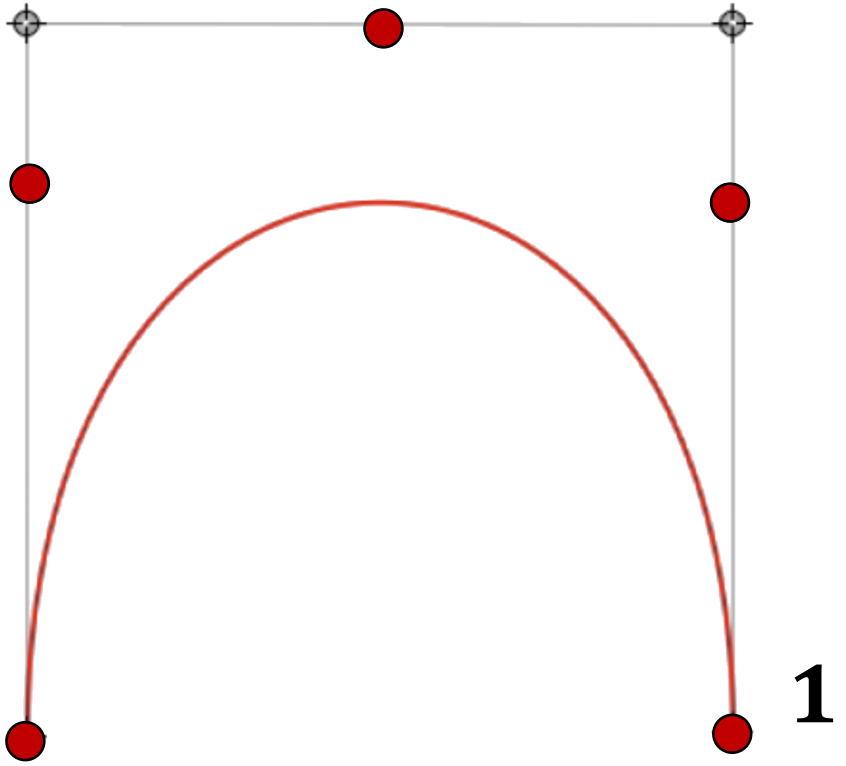

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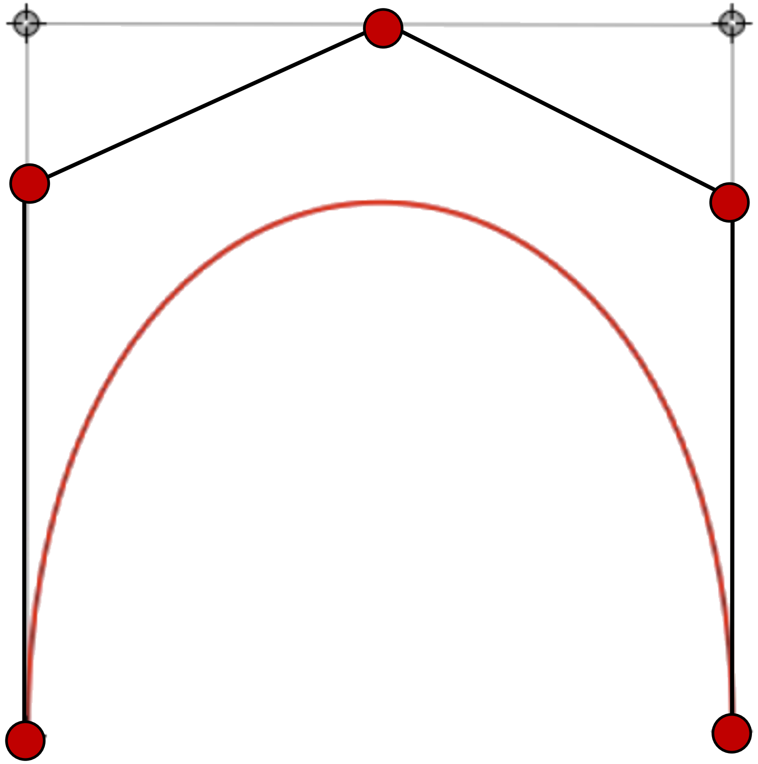
Degree elevation: Example



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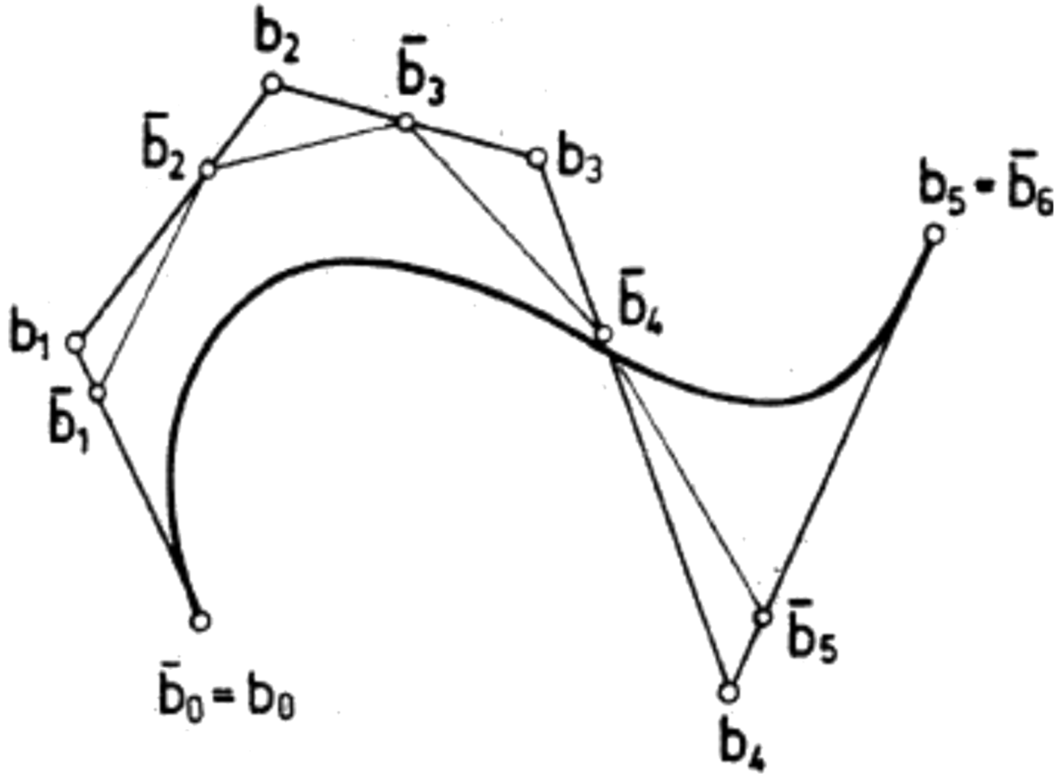
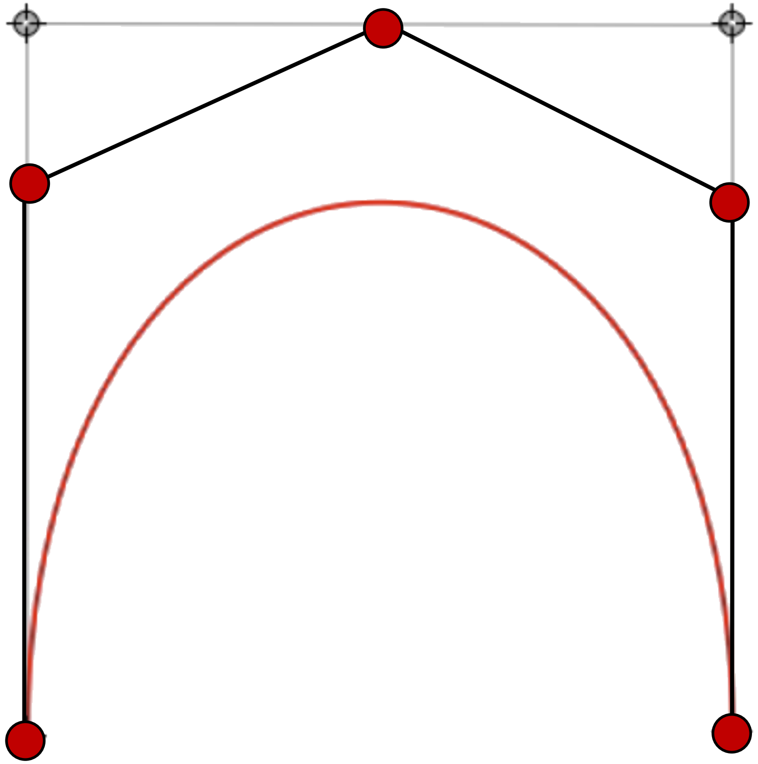
Degree elevation



For repeated degree elevation, the Bézier polygon converges to the Bézier curve. (slow convergence)

Degree elevation



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$$j = 1, \dots, n$$

Bézier Curves

Subdivision

Subdivision

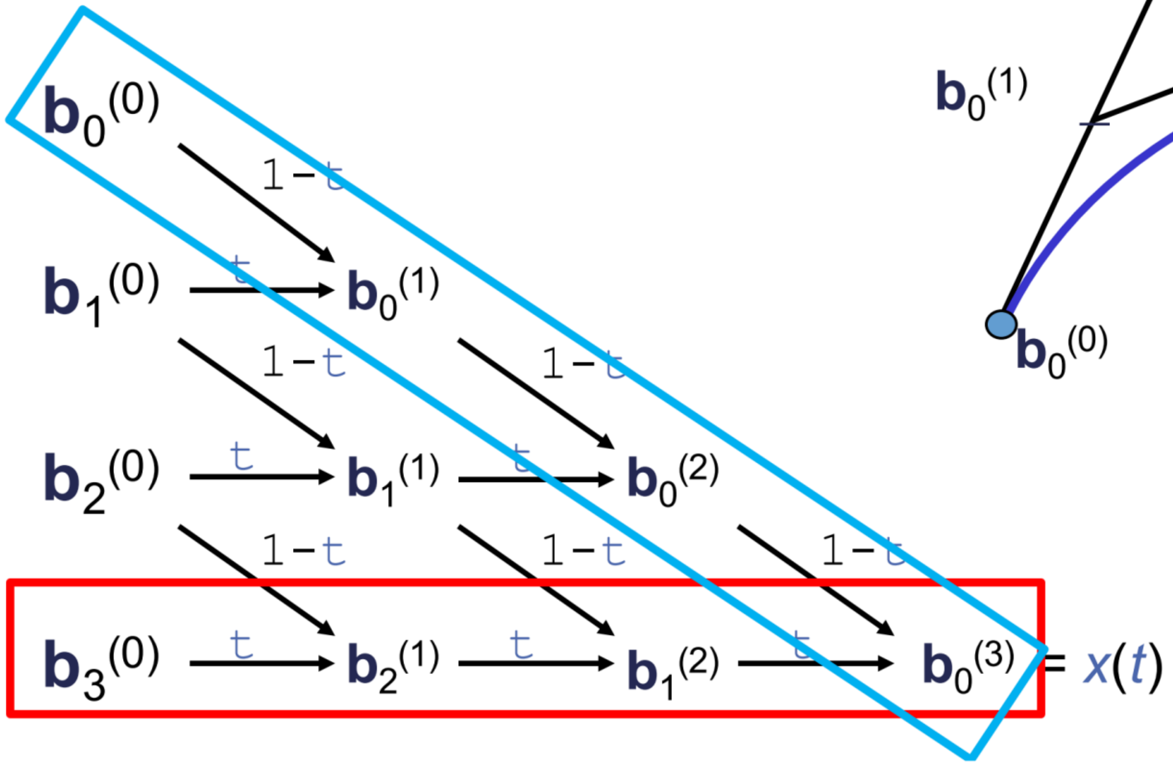
- **Given:** $b_0, \ldots, b_n \to x(t), t \in [0,1]$

- **Wanted:** $b_0^{(1)}, \ldots, b_n^{(1)} \to x^{(1)}(t),$

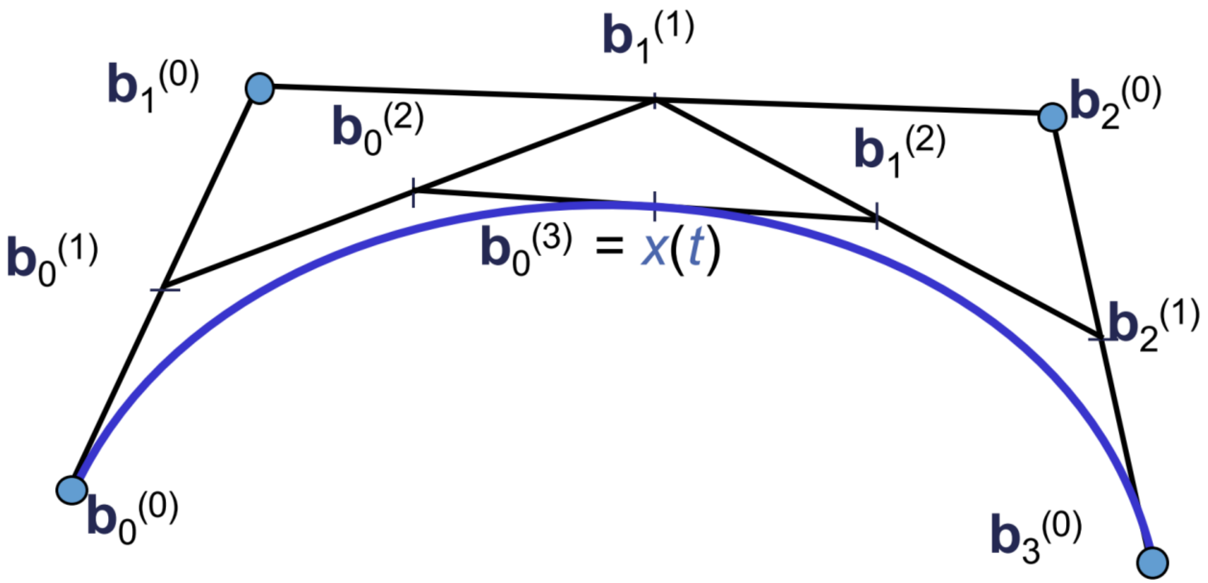
  $b_0^{(2)}, \ldots, b_n^{(2)} \to x^{(2)}(t),$

  with $x = x^{(1)} \cup x^{(2)}$

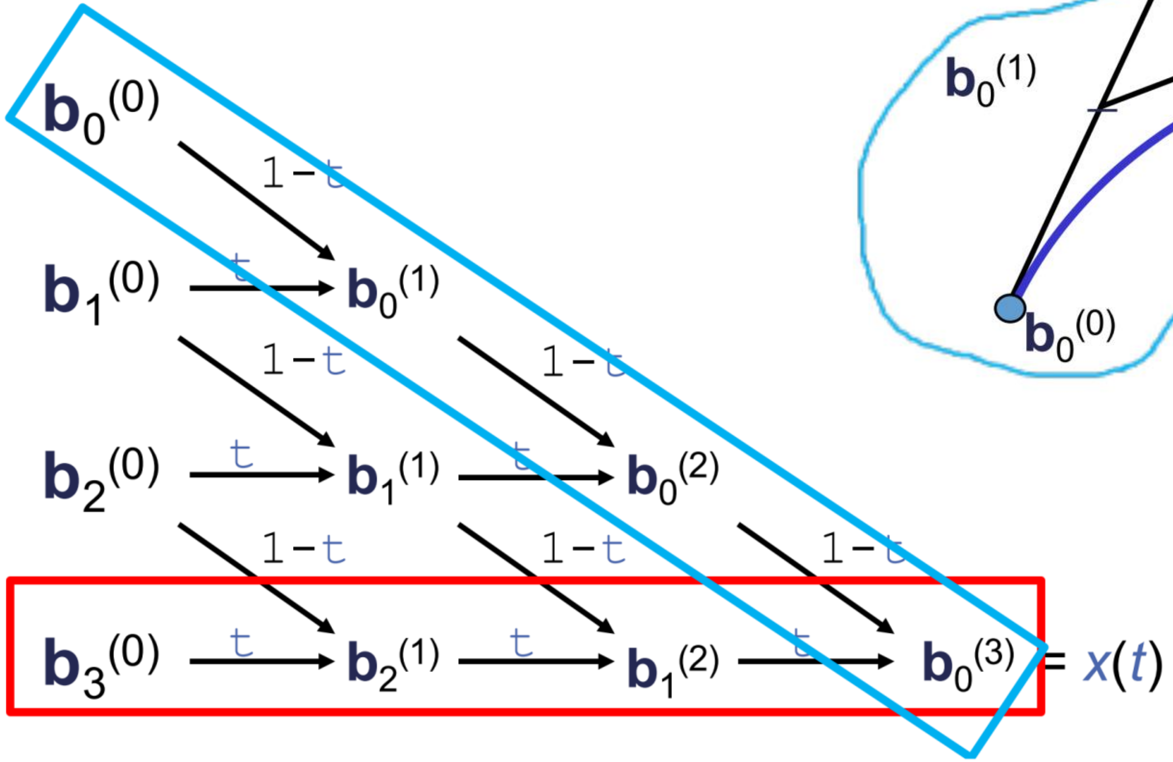
Subdivision: Example



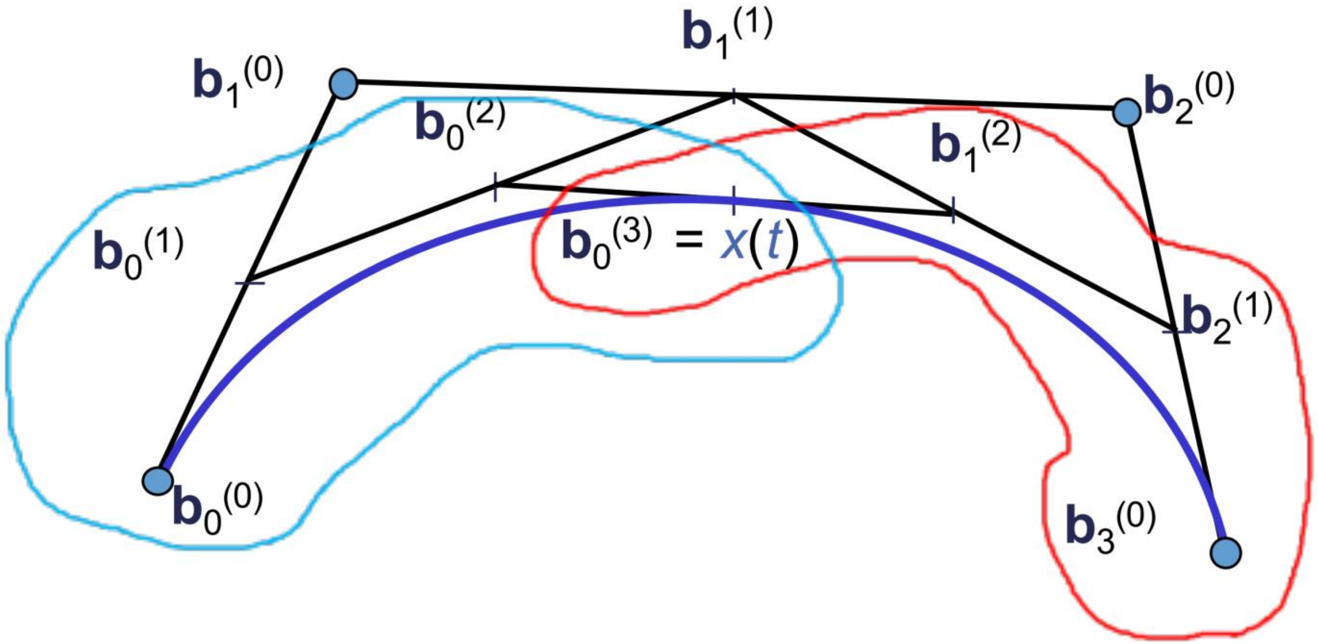
de Casteljau scheme



Subdivision: Example



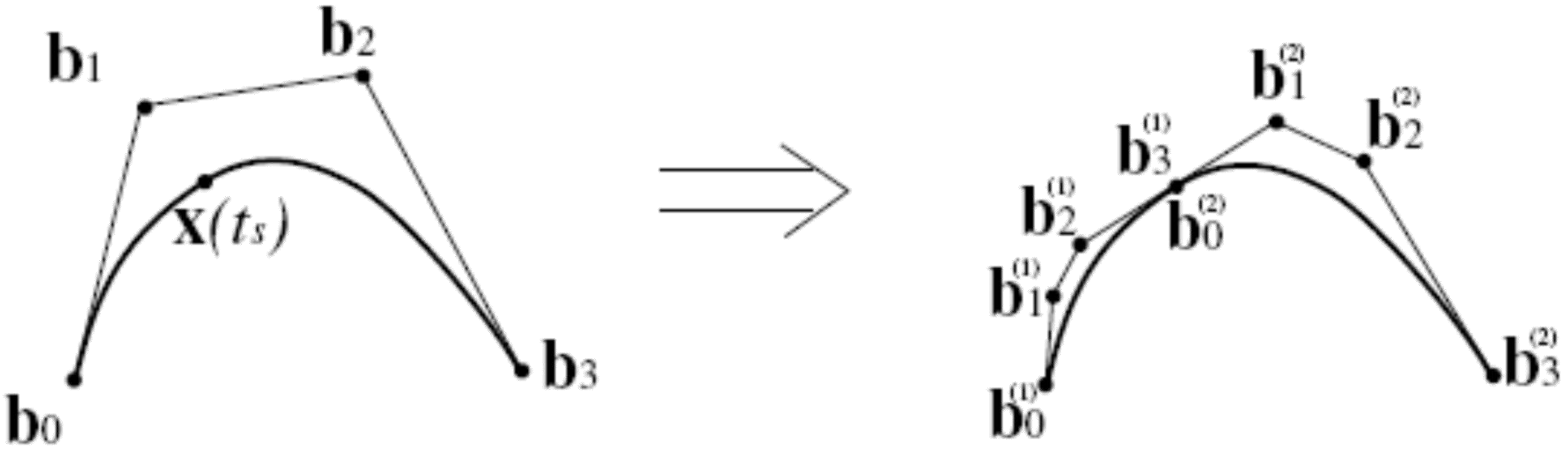
de Casteljau scheme



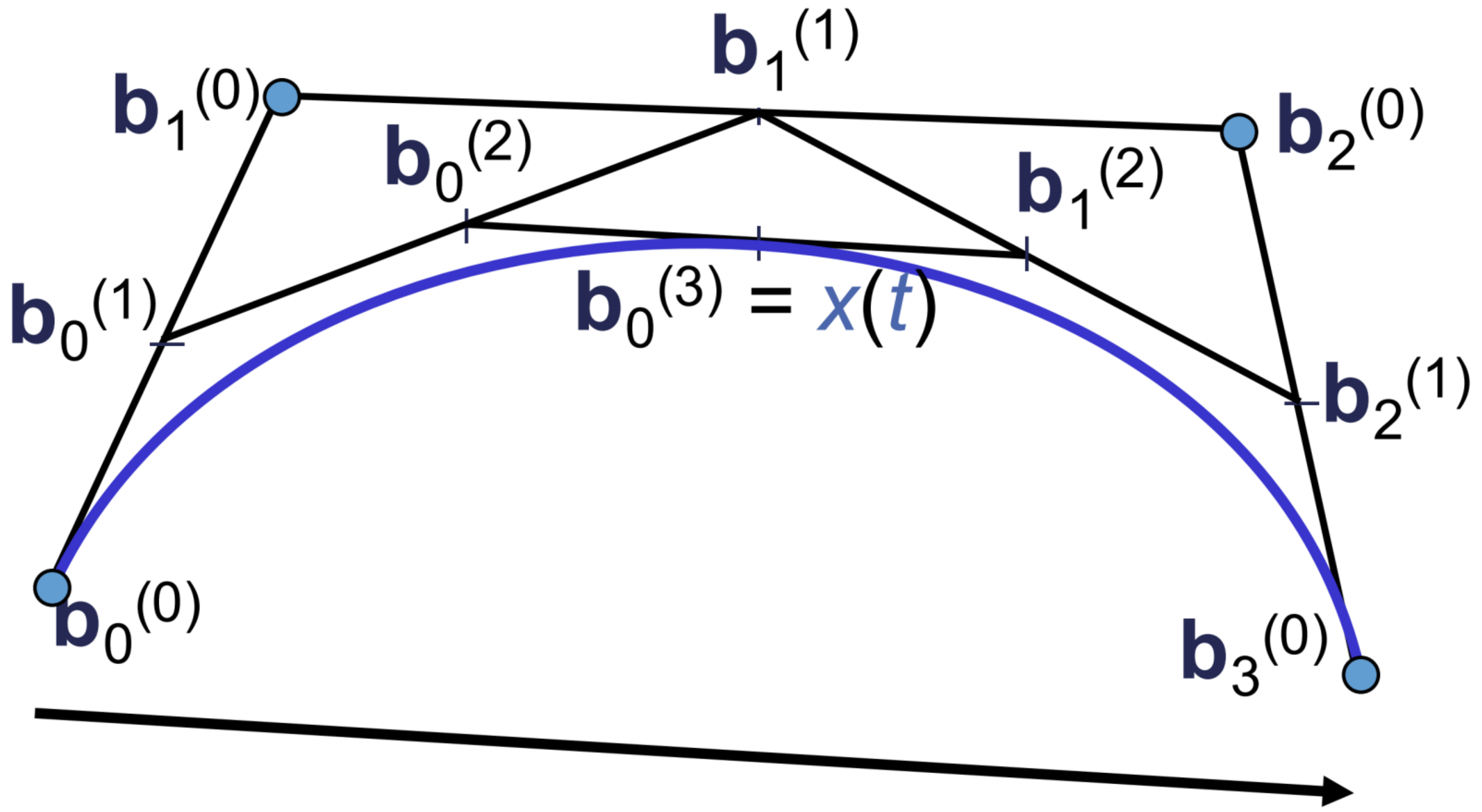
Subdivision

Solution: $b_i^{(1)} = b_0^i, b_i^{(2)} = b_0^{n-i}$ for $i = 0, \ldots, n$

That means that the new points are intermediate points of the de Casteljau algorithm!

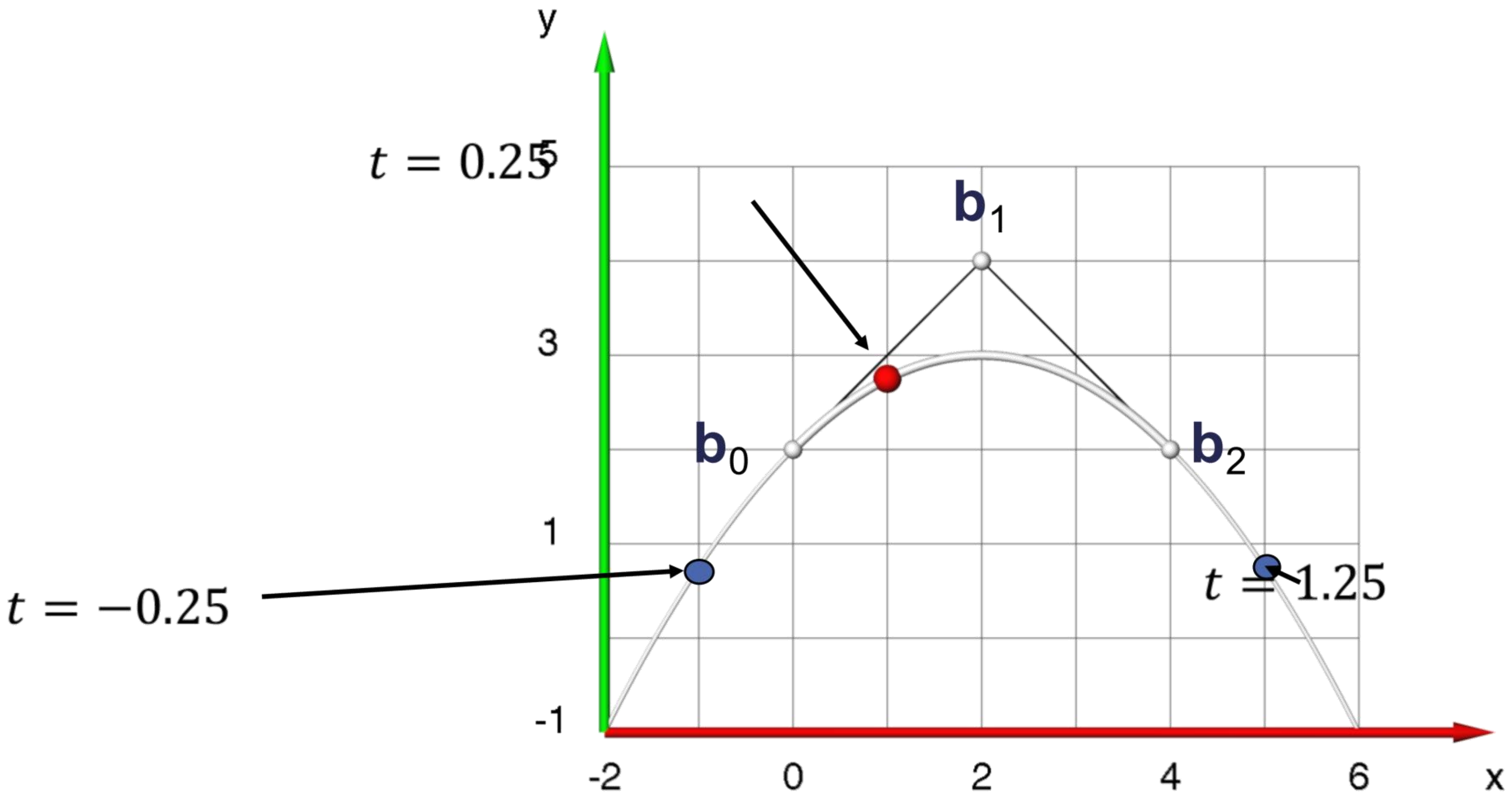


Curve range



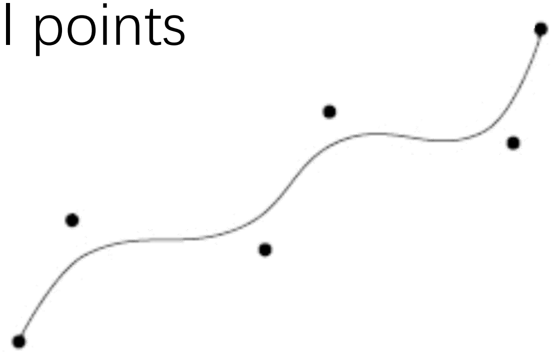
parameterization: $t \in [0,1]$

Curve range



Summary & Outlook

- **Bézier curves and curve design**
  - The rough form is specified by the position of the control points
  - Results: smooth curve approximating the control points
  - Computation / Representation:
    - de Casteljau algorithm
    - Bernstein form



  - Problems:
    - High polynomial degree
    - Moving a control point can change the whole curve
    - Interpolation of points
    - →**Bézier splines**

Matrix representations

(common in software implementations)

Homogeneous coordinates

$$[P] = \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & y_n & z_n & 1 \end{pmatrix}$$

Transformations

- Basic representation $[P^*] = [P][T]$

  $[P^*]$ is the new coordinates matrix

  $[P]$ is the original coordinates matrix, or points matrix

  $[T]$ is the transformation matrix

Transformations

- Translation (2D example)

$$[T_t] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

$$[P^*] = [P][T_t]$$

Transformations

- Basic representation $[P^*] = [P][T]$

  $[P^*]$ is the new coordinates matrix

  $[P]$ is the original coordinates matrix, or points matrix

  $[T]$ is the transformation matrix

$$[P] = \begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \\ \dots & \dots & \dots \\ x_n & y_n & 0 \end{pmatrix}$$

Transformations

- Uniform scaling

$$[T] = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Non-uniform scaling

$$[T] = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation 2D

$$\left.\begin{aligned} x &= r\cos\alpha \\ y &= r\sin\alpha \end{aligned}\right\} \text{Original coordinates of point } P$$

$$\left.\begin{aligned} x^* &= r\cos(\alpha+\theta) \\ y^* &= r\sin(\alpha+\theta) \end{aligned}\right\} \text{The new coordinates}$$

$$[x^* \quad y^* \quad 0 \quad 1] = [x \quad y \quad 0 \quad 1]\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation about an arbitrary axis (2D)
  - Translate the fixed axis so it coincides with z-axis

    → apply to object
  - Rotate object about the axis
  - Translate object back

$$[P^*] = [P][T_t][T_r][T_{-t}]$$

Transformations

- Rotation about an arbitrary axis (2D)

Step 1: Translate the fixed axis so it coincides with z-axis

Step 2: Rotate object about the axis

Step 3: Translate the fixed axis back to the original position

$$[P^*] = [P][T_t][T_r][T_{-t}]$$

Transformations

- Scaling with an arbitrary point (x, y)

$$[P^*] = [P][T_t][T_s][T_{-t}]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ x - sx & y - sy & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation about an arbitrary point (x, y)

$$[T_{\text{cond}}] = [T_t][T_s][T_{-t}]$$

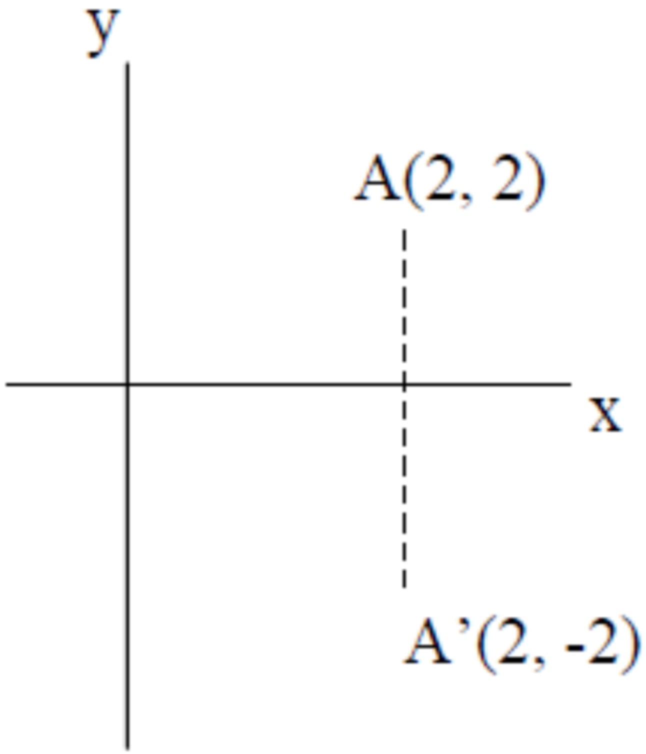
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

Transformations

- Mirroring about x-axis (negative scaling along y-axis)

$$[P^*] = [2 \quad 2 \quad 0 \quad 1]\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= [2 \quad -2 \quad 0 \quad 1]$$



Transformations

- Mirroring about arbitrary axis

  - Translate line to pass through origin
  - Rotate axis to coincide with x-axis
  - Mirror about x-axis
  - Rotate back
  - Translate back to original position

$$[P^*] = [P][T_t][T_r][T_m][T_{-r}][T_{-t}]$$

Transformations

- Rotation about coordinates axes (3D)

$$[T_{rz}] = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[T_{rx}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[T_{ry}] = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation θ about an arbitrary axis (3D)
    1. Translate the given line so that it will pass through the origin
    2. Rotate about the x-axis so that the line lies in the xz-plane (angle α)
    3. Rotate about the y-axis so that the line coincides with the z-axis (angle ϕ)
    4. Rotate the geometric object about the z-axis (angle θ – given rotation angle)
    5. Reverse of step 3
    6. Reverse of step 2
    7. Reverse of step 1

$$[P^*] = [P][T_t][T_r]_\alpha[T_r]_\phi[T_r]_\theta[T_r]_{-\phi}[T_r]_{-\alpha}[T_{-t}]$$

Alternatively you can use Quaternions!

Bézier Curves

- Cubic Bézier curves $$f(t) = P_0 B_0^{(3)} + P_1 B_1^{(3)} + P_2 B_2^{(3)} + P_3 B_3^{(3)}$$

$$B_0^{(3)}(t) = \frac{3!}{0!\,3!} t^0 (1-t)^3 = (1-t)^3$$

$$B_1^{(3)} = \frac{3!}{1!\,2!} t^1 (1-t)^2 = 3t(1-t)^2$$

$$B_2^{(3)} = \frac{3!}{2!\,1!} t^2 (1-t)^1 = 3t^2(1-t)$$

$$B_3^{(3)} = \frac{3!}{3!\,0!} t^3 (1-t)^0 = t^3$$

Bézier Curves

- Cubic Bézier curves
  $$f(t) = P_0 B_0^{(3)} + P_1 B_1^{(3)} + P_2 B_2^{(3)} + P_3 B_3^{(3)}$$
- In Matrix form:
  - -The curve
    $$f(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$
  - -The tangent
    $$f'(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

B-spline Curves (to be covered later)

- Uniform cubic B-Spline curve

$$f_i(t) = \frac{1}{6}[t^3 \quad t^2 \quad t \quad 1]\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}\begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix}$$

B-spline Curves (to be covered later)

- Other splines:
  - Catmull-Rom
  - Cardinal splines
  - Tensioned splines

$$f_i(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \frac{1}{2}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix}$$

$$\begin{pmatrix} -a & 2-a & a-2 & a \\ 2a & a-3 & 3-2a & -a \\ -a & 0 & a & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{6}\begin{pmatrix} -a & 12-9a & 9a-12 & a \\ 2a & a-3 & 18-15a & -a \\ -3a & 0 & 3a & 0 \\ 0 & 6-2a & a & 0 \end{pmatrix}$$